J. Fluid Mech. (1971), *vol.* 46, *part* 2, *pp.* 337–355
Printed in Great Britain

Hydromechanics of swimming propulsion. Part 1. Swimming of a two-dimensional flexible plate at variable forward speeds in an inviscid fluid

By T. YAO-TSU WU
California Institute of Technology,
Pasadena, California

(Received 21 July 1970)

The most effective movements of swimming aquatic animals of almost all sizes appear to have the form of a transverse wave progressing along the body from head to tail. The main features of this undulatory mode of propulsion are discussed for the case of large Reynolds number, based on the principle of energy conservation. The general problem of a two-dimensional flexible plate, swimming at arbitrary, unsteady forward speeds, is solved by applying the linearized inviscid flow theory. The large-time asymptotic behaviour of an initial-value harmonic motion shows the decay of the transient terms. For a flexible plate starting with a constant acceleration from at rest, the small-time solution is evaluated and the initial optimum shape is determined for the maximum thrust under conditions of fixed power and negligible body recoil.

1. Introduction

Aquatic animals propelling themselves in water, or in other liquid media, span a wide range in their sizes and speeds. Large cetaceans, such as porpoises and whales, may have lengths from 2 to 30 m, and can swim at cruising speeds of from 6 to 10 m/s (Lang & Pryor 1966). Microscopic organisms such as paramecia and spermatozoa, ranging from 300 μm down to 50 μm in length with length–diameter ratio from 20 to 100, can swim at speeds from 1000 to 80 μm/s. In between these two extremities there are many species of fishes and aquatic animals of various sizes. Based on the characteristic length l of a body moving at velocity U in a liquid of kinematic viscosity ν, the Reynolds number, $R = Ul/\nu$, measures the relative magnitude of the time average of inertial stress to viscous stress. The value of R is of order 10^8 for the most rapid cetaceans, 10^6 for migrating fishes, 10^5–10^3 for a great variety of fishes, about 10^2 for tadpoles, down to about 1 for *Turbatrix*, 10^{-3} or less for paramecia and spermatozoa (Gray 1968, p. 437), and to the extreme of 10^{-6} or less for bacteria. Thus, the Reynolds number R covers practically the entire range of interest known to hydrodynamicists. Lighthill (1969) has given an excellent survey of the hydromechanics of aquatic animal propulsion, which has elucidated both the zoological and hydromechanical aspects of the subject.

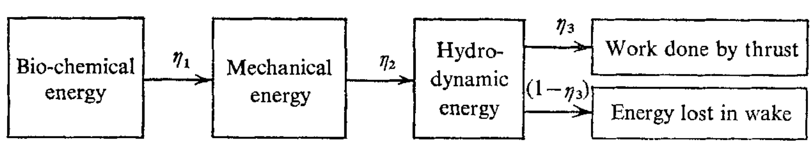
Although R may vary greatly from case to case, the most effective movements of swimming propulsion employed by a large number of aquatic animals of drastically different sizes have been observed to differ very little from an undulatory motion of the body, in the form of a transverse wave propagating along the body from head to tail. A great majority of many species of fishes can be singled out as a pre-eminent class of this mode of propulsion. The remarkable performance of some cetaceans (dolphin, porpoises, whales, etc.) and some well-known game fish families (tuna, wahoo, marlin, swordfish, etc.), using strong tails of large aspect ratio, is only a variation of this basic undulatory mode. In the world of micro-organisms, an enormous variety of creatures, ranging from minute bacteria, larger but still primitive protozoa, to higher level spermatozoa, have been observed to employ either uniformly propagating transverse waves, or whip-like waves, or helical waves along slender flagella as principal means of propulsion. The basic transverse wave mode thus seems to be little affected by the Reynolds number over such a wide range. However, the fundamental principles underlying the hydromechanics of swimming propulsion do become very different for large or small values of the Reynolds number.

For Reynolds number large, the swimming propulsion depends primarily on the inertial effect, since the flow outside a thin boundary layer next to the body surface is irrotational. Viscosity of the fluid is unimportant except in its role of generating the vorticity shed into the wake, and of producing a thin boundary layer, and hence a skin friction at the body surface. As the body performs an undulatory wave motion and attains a forward momentum, the propulsive force pushes the fluid backward with a net total momentum equal and opposite to that of the action, while the frictional resistance of the body gives rise to a forward momentum of the fluid by entraining some of the fluid surrounding the body. The momentum of reaction to the inertial forces is concentrated in the vortex wake due to the small thickness and amplitude of the undulatory trailing vortex sheet; this backward jet of fluid expelled from the body can, however, be counterbalanced by the momentum in response to the viscous drag. When a self-propelled body is cruising at a constant speed, the forward and backward momenta exactly balance; they can nevertheless be evaluated separately. This mechanism of swimming motion at large Reynolds numbers has been elucidated by von Kármán & Burgers (1943) for the simple case of a rigid plate in transverse oscillation. Swimming of slender fish has been treated by Lighthill (1960); and the waving motion of a two-dimensional flexible plate has been calculated by Wu (1961).

In the other extremity, movements of microscopic bodies always correspond to small Reynolds numbers. The propulsion in this range depends almost entirely on the viscous stresses, since the inertial forces are then extremely small, except possibly for the motions at very high frequencies. Oscillatory motions in a viscous fluid were discussed as early as 1851 by Stokes. Various studies of the swimming of microscopic organisms have been led by Taylor (1951, 1952*a*, *b*), who discussed the propulsion of a propagating, monochromatic, transverse wave along a sheet immersed in a very viscous fluid, and later evaluated the action of waving cylindrical tails of microscopic organisms. Further studies in this field have been contributed by Hancock (1953), Gray & Hancock (1955), Reynolds (1965) and Tuck (1968).

Aside from the mode of propagating transverse waves, there are still other kinds of body motions, such as (*a*) actual ejecting of liquid as employed by squids and octopus, (*b*) progressive waves along fringe belts as used by some flat fishes, and waving motion produced by bending a large number of dense tassels underneath a star fish, (*c*) squirming motion by changing the body shape of a tail-less object in slow motion through a viscous fluid, (*d*) ciliated propulsion of numerous micro-organisms by waving movements of a large number of cilia attached to the body surface. Problem (*a*) has been discussed by Siekmann (1963), and (*c*) has been analysed by Lighthill (1952). Close resemblance between the movements of cilia and flagella has been contended by some investigators. The problem of self-propulsion of a deformable body in a perfect fluid, having absolutely no viscosity, has been discussed by Saffman (1967).

Hydrodynamics of swimming is only a part of the whole problem. From the viewpoint of bioengineering, the entire process begins with the biochemical energy stored in the swimming being, which can be converted, with efficiency η_1, into mechanical energy for maintaining the body motion; the latter is in turn transformed, with efficiency η_2, into hydrodynamic energy for swimming. A part (fraction η_3 say) of the hydrodynamic energy is spent as the useful work done by the thrust, which balances the work done by frictional drag, and the remaining part becomes the energy lost, or dissipated, in the flow wake. It is



in the effort of keeping a self-contained balance of energy that some apparently astonishing observations have been reported. For example, Johannessen & Harder (1960) reported several impressively high speeds (about 20 to 22 knots) attained by porpoises, killer whales and black whales. The boundary layer over a rigid, smooth surface of a similar body in this Reynolds number range is definitely turbulent. If the skin friction is evaluated on this basis, then the power required to maintain such high speeds would violate by severalfold the rule of thumb in biology that a pound of strong muscle can deliver only up to 0·01 horsepower. More recently, the speed of porpoises has been investigated carefully, under well-controlled conditions, by Lang *et al.* (1963, 1966). Another interesting study is that of migratory salmon by Osborne (1960). According to this careful investigation, a detailed estimate again led to one of two conclusions: either (*a*) these creatures have a much smaller drag than could be achieved with similar, rigid bodies, or (*b*) the power output per gram of muscle is much larger than that observed from physiological experiments on warm-blooded animals. This is known as the paradox of Gray (1948, 1949). These puzzling alternatives have stimulated fluid mechanists to explore various other possibilities, such as the effect of compliant skin, and the effects of mucous surface and additives on frictional drag, studies of the former being so far inconclusive. The subject of drag reduction by long-polymer additives has been under active development.

Also, the recent study of Lang & Daybell (1963) has given partial explanation to Gray's dilemma.

The present study is devoted to several hydromechanical problems of swimming propulsion. In part 1, the main features and advantages of the undulatory mode with a transverse wave progressing along a planar or slender body are discussed from the principle of energy conservation. The two-dimensional oscillating-airfoil theory appears to be of value in fish hydromechanics as a first approximation to the propulsion of lunate tail (percomorph fish, fast sharks, cetacean mammals and variants: Lighthill 1969) with caudal fins of high aspect ratio. If this type of theory is applied to the flapping flight of birds, it is more realistic to consider the forward velocity of the wing to be non-uniform, since it is known that the wing movements, in up-and-down strokes, have very large backward-and-forward components (Gray 1968). Furthermore, such a general theory may also have applications to artificial propulsive devices, such as the vertical-axis propeller (Voith–Schneider type, whose blades move relative to fluid with variable velocity and pitch), and may be particularly useful in the control theory for hydrofoils and other devices when the transient behaviour is of importance. With these future applications in view, the general problem of a two-dimensional flexible plate moving with a variable forward velocity is solved based on the linearized wing theory. The small-time solution of a flexible plate starting with a constant acceleration from at rest is evaluated, and the optimum shape in the initial stage is determined for the maximum thrust under the condition of fixed power and small body recoil. The problem of optimum movement of a rigid wing and the general optimum shape of a flexible plate in harmonic time motion will be treated in part 2 of this study. In part 3 the effect of a vortex sheet shed by side fins of a slender fish in swimming will be discussed together with the optimum movement.

2. Thrust; energy balance

In order to understand why the motion of a transverse wave progressing along the body is desirable for swimming propulsion, we consider the energy balance for the typical case of a flexible planar body of negligible thickness, performing an arbitrary unsteady motion of small amplitude, achieving in time t a rectilinear forward velocity $U(t)$ through a fluid which is otherwise at rest. We choose a Cartesian co-ordinate system (x, y, z) fixed at the mean position of the body, with the stretched plan form of the body lying in the $y = 0$ plane and with the free-stream velocity $U(t)$ pointing in the positive x direction. The body motion can be written generally as

$$y = h(x, z, t) \quad (x, z \in S), \tag{1}$$

where S is the stretched plan form of the body (when h vanishes identically), h is an arbitrary function of x, z, and t, with $|\partial h/\partial t|$ and swimming velocity U assumed to be sufficiently small for the flow to be regarded as incompressible, and with $|\partial h/\partial x|$ and $|\partial h/\partial z|$ assumed also small enough to justify the linear theory.

The Reynolds number $R = Ul/\nu$, based on the velocity U and body length l (in the streamwise direction), is taken to be so large that the boundary layer is thin and the inertial effects can be evaluated under the inviscid flow assumption. Then the boundary condition requiring the normal component of velocity relative to the solid surface to vanish prescribes the y component of the flow velocity at the planar surface as

$$v(x, \pm 0, z, t) \equiv V(x, z, t) = \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} \quad (x, z \in S). \tag{2}$$

The planar body may admit sharp leading and trailing edges. When the latter kind is present, we shall impose, as usual, the Kutta condition that the flow speed and pressure are required to be continuous at sharp trailing edges. The following discussion is also applicable to plane flows, say in the x, y plane, in which case the dependence on z simply drops out, and all the quantities will then refer to a unit span in the z direction.

The thrust (positive when directed in the negative x direction) acting on the body, based on the inviscid linear theory, is given by the integration of the pressure component in the forward direction,

$$T = T_p + T_s = \int_S (\Delta p)\frac{\partial h}{\partial x}dS + \int_{\text{L.E.}} F_s(x, z, t)\,dz, \tag{3}$$

where (Δp) denotes the pressure difference across the flexible plate,

$$\Delta p = p(x, -0, z, t) - p(x, +0, z, t),$$

F_s is the singular force per unit arc length along the leading edge due to the leading-edge suction, and the last integral is evaluated along the leading edge $z = b(x)$. The power required to maintain the motion is equal to the time rate of work done by the plate against the reaction of the fluid in the direction of the transverse plate motion,

$$P = -\int_S (\Delta p)\frac{\partial h}{\partial t}dS. \tag{4}$$

The third quantity of interest is the mechanical energy imparted to the fluid in unit time, which in this inviscid flow is equal to the time rate of work done by the pressure over the body surface, or

$$E = -\int_S (\Delta p)\, V(x,z,t)\,dS - T_s U. \tag{5}$$

The above three quantities satisfy the principle of conservation of energy which asserts that the power input P is equal to the rate of work done by the thrust, TU, plus the kinetic energy E lost to the fluid in unit time,

$$P = TU + E. \tag{6}$$

If the viscous effects are further taken into account, then the thrust T must include the viscous drag due to skin friction and the energy loss must contain the viscous dissipation.

On physical grounds it can be inferred that the energy loss E is non-negative in several cases of broad interest. One such case is the periodic body movement with constant forward velocity,

$$U = \text{const.}, \quad h(x,z,t) = h_1(x,z) \exp(j\omega t) \quad (x,z \in S), \tag{7}$$

where $j = \sqrt{-1}$ is the imaginary unit for the period time motion, $h_1(x,z)$ may generally be complex with respect to j, and h is to be interpreted by its real part. After the transient stage is over, the kinetic energy imparted to the fluid will be largely confined in the wake which contains the trailing vortex sheet and is lengthening at the rate U. Therefore E, or at least its time average, cannot be negative. (A mathematical proof of this statement has been given for slender bodies by Lighthill (1960) and for waving plates in plane flows by Wu (1961).) Another example is when the body starts to swim from a state of rest:

$$U = U(t), \quad h = h(x,z,t) \quad (t > 0), \tag{8}$$

while U, h, and the components of the perturbation velocity (u, v, w) all vanish for $t \leqslant 0$. In this case any disturbance generated in the flow must correspond to a gain of kinetic energy of the fluid (see § 6).

The following discussion will be based on the presumption $E \geqslant 0$. Under this condition we have, by (6),

$$P \geqslant TU \quad \text{if} \quad E \geqslant 0. \tag{9}$$

P, however, need not be positive definite. When P is negative, energy is transferred out of the fluid (like a turbine); then $T < 0$ according to (9), indicating that there must be an inertial drag acting on the body. Forward swimming is possible only when the thrust $T > 0$, large enough to overcome the viscous drag; then $P > 0$, and hence a power is required to maintain the motion. Now, from (3) it is seen that a sufficient condition for producing a positive thrust is satisfied if Δp and $\partial h/\partial x$ are everywhere of the same sign, for the suction force F_s in forward movement is never negative. In view of the inequality (9) and the expression (4) for P, Δp and $\partial h/\partial t$ cannot have also the same sign everywhere on S. Suppose, as a qualitative picture, $\partial h/\partial x$ and $\partial h/\partial t$ are everywhere opposite in sign, then clearly h represents a transverse wave propagating towards the tail (see figure 1).

To investigate further the qualitative features of such periodic waving motions it suffices to consider the case of simple harmonic form (7), since, for arbitrary time dependence, all linear effects (such as the pressure, lift, moment) can be obtained by the Fourier synthesis; and, as for the quadratic effects such as T, P, and E, it can be seen that, in their time averages, the components with different multiple-frequencies are not coupled. In fact, consider two functions:

$$g(\mathbf{x},t) = \text{Re}\left[\sum_n g_n(\mathbf{x}) \exp(j\omega_n t)\right], \quad h(\mathbf{x},t) = \text{Re}\left[\sum_n h_n(\mathbf{x}) \exp(j\omega_n t)\right], \tag{10}$$

where Re denotes the real part, the time average of gh is

$$\overline{gh} \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T g(\mathbf{x},t)\, h(\mathbf{x},t)\, dt = \tfrac{1}{2}\,\text{Re}\left[\sum_n g_n(\mathbf{x})\, h_n^*(\mathbf{x})\right], \tag{11}$$

where h^* is the complex conjugate of h (with respect to j). This result is readily extended to the integral form when g, h are expressed by integrals over a continuous spectrum.

Returning to the waving motion, we consider the fundamental form

$$h = \mathrm{Re}\,[h_1(x,z)\exp(j(\omega t - kx))] \quad (x,z \in S), \tag{12}$$

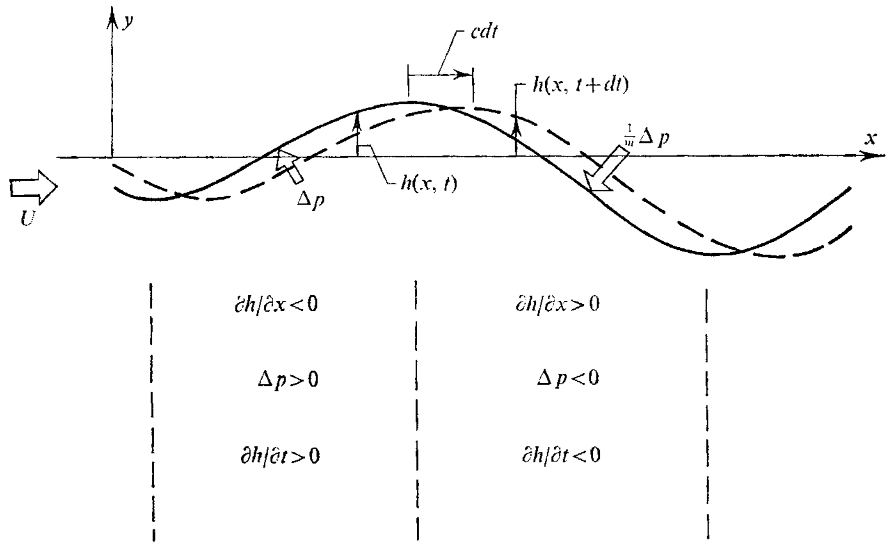


FIGURE 1. Consideration of energy conservation indicates that, in forward swimming, transverse movements of body wave propagate not only backward (from head to tail) with velocity c, but also backward relative to the fluid, since $c \geqslant U$.

which represents a simple wave propagating along the planar body in the streamwise direction with phase velocity $c = \omega/k$ and amplitude $|h_1(x,z)|$. Substituting (12) in (3) and (4), and taking the time average, we obtain

$$\overline{T}_p = \frac{k}{2}\mathrm{Re}\int_S (\Delta p_1)\left(jh_1^* + \frac{1}{k}\frac{\partial h_1^*}{\partial x}\right)\exp(jkx)\,dS, \tag{13a}$$

$$\overline{P} = \frac{\omega}{2}\,\mathrm{Re}\int_S (\Delta p_1)(jh_1^*)\exp(jkx)\,dS, \tag{13b}$$

where $(\Delta p_1) = (\Delta p)\exp(-j\omega t)$, is independent of t as a result of the linearized theory. Since the thrust T_s due to the leading-edge suction is always non-negative, it follows from inequality (9) that

$$\overline{P} \geqslant U\overline{T} \geqslant U\overline{T}_P, \tag{14}$$

provided $\overline{E} \geqslant 0$. Consequently, if $\partial h_1/\partial x = 0$, or if $|\partial h_1/\partial x| \ll |kh_1|$, then from (13) and (14) we immediately have

$$c = \omega/k \geqslant U. \tag{15}$$

This result shows that not only is a progressive transverse wave desirable, but also its phase velocity must be greater than U (under the stated conditions), in order to achieve a given swimming velocity U. This qualitative feature remains true for a wide class of amplitude function $h_1(x,z)$.

3. Swimming of a waving plate with variable forward velocity in plane flow

Though the flow around swimming fish is certainly three-dimensional, the theory of two-dimensional swimming motion is still of considerable interest, since it can be applied to estimate the propulsion of a tail of large aspect ratio of some species of cetaceans and also various fishes such as scombroids and the faster sharks, or even the propulsion of wing flappings of migrating birds. We derive in the following the main features of swimming with arbitrary forward velocity in plane flows.

Here we consider the incompressible plane flow of an inviscid fluid past a flexible plate of zero thickness, spanning from $x = -1$ to $x = 1$, and performing a waving motion of the general form

$$y = h(x,t) \quad (-1 < x < 1, \quad t > 0), \tag{1$'$}$$

h being again an arbitrary, continuous function of x, t, and assumed to be always small. (The effect of small thickness of a planar body of this type is regarded as secondary, and can be estimated separately.) The motion starts at $t = 0$ from a uniform state; the free-stream velocity $U(t)$ may depend on t. Let u and v again denote respectively the x and y component of the perturbation velocity. We introduce the Prandtl acceleration potential,

$$\phi(x,y,t) = (p_\infty - p)/\rho, \tag{16}$$

where p_∞ is the pressure at infinity and ρ is the fluid density. In the linear theory of this incompressible irrotational flow, p, and hence also ϕ, is a harmonic function of x, y for all t. A harmonic function $\psi(x,y,t)$ conjugate to ϕ may be defined by $\phi_x = \psi_y$, $\phi_y = -\psi_x$, where the subscripts x and y denote differentiations. Hence the complex acceleration potential $f = \phi + i\psi$ and the complex velocity $w = u - iv$ are analytic functions of the complex variable $z = x + iy$ for all real t. (We borrow the notation w and z for this different purpose in this section). f and w are related by the linearized Euler's equation of motion,

$$\frac{\partial f}{\partial z} = \frac{\partial w}{\partial t} + U(t)\frac{\partial w}{\partial z}. \tag{17}$$

The linearized boundary conditions are

$$v(x, 0\pm, t) \equiv V(x,t) = h_t + Uh_x \quad (-1 < x < 1), \tag{18}$$

$$-\frac{\partial \psi}{\partial x} = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) V \quad (y = 0\pm, \quad |x| < 1) \tag{19}$$

$$\phi(x,0,t) = 0 \quad (|x| > 1), \tag{20}$$

$$|f(1,t)| < \infty \quad \text{for all } t, \tag{21}$$

$$f(z,t) \to 0 \quad \text{as} \quad |z| \to \infty; \quad w(z,t) \to 0 \quad \text{as} \quad z \to -\infty. \tag{22}$$

Here, condition (19) follows from the imaginary part of (17) and condition (18); condition (20) follows from the fact that ψ is even, and hence ϕ is odd in y; (21)

is the Kutta condition for the flow at the trailing edge $z = 1$. Condition (22) for w may also be specified as $|z| \to \infty$, $|\arg z| > 0$, i.e. as $z \to \infty$ in the region excluding the trailing vortex sheet.

Integration of (17) to obtain the boundary value of ψ on the plate can be done by using the method of characteristics. However, with variable $U(t)$, it is more convenient to make use of the Laplace transform method. We first introduce the variable

$$\tau(t) = \int_0^t U(t)\,dt \quad (t > 0), \tag{23}$$

and assume its inverse function $t = t(\tau)$ is unique so that $U = U(t(\tau))$ is a one-valued function of τ, this being the case so long as the swimming proceeds in one direction. Regarding w and f as functions of z and τ, (17) becomes

$$\frac{\partial F}{\partial z} = \frac{\partial w}{\partial \tau} + \frac{\partial w}{\partial z}, \tag{24}$$

where $$F(z,\tau) = f(z,\tau)/U(\tau) = \Phi(x,y,\tau) + i\Psi(x,y,\tau). \tag{25}$$

Application of the Laplace transform

$$\tilde{F}(z,s) = \int_0^\infty \exp(-s\tau)\,F(z,\tau)\,d\tau \quad (\mathrm{Re}\,s > 0) \tag{26}$$

to (24), under zero initial conditions, yields

$$\frac{d\tilde{F}}{dz} = \left(\frac{d}{dz} + s\right)\tilde{w}. \tag{27}$$

Integrating this equation from $z = -\infty$, using conditions (22), and expressing $\tilde{F}$ in terms of $\tilde{w}$, and vice versa, we obtain its imaginary part as

$$\tilde{\Psi}(x,y,s) = -\tilde{v}(x,y,s) - s\int_{-\infty}^{x} \tilde{v}(x_1,y,s)\,dx_1, \tag{28a}$$

or $$\tilde{v}(x,y,s) = -\tilde{\Psi}(x,y,s) + s\int_{-\infty}^{x} \exp\,(s(x_1 - x))\,\tilde{\Psi}(x_1,y,s)\,dx_1. \tag{28b}$$

On the plate, with $y = 0\pm$ and $|x| < 1$, $\tilde{v}(x, 0\pm, s) = \tilde{V}(x,s)$, which is the Laplace transform of (18). Application of this condition to (28a) yields

$$\tilde{\Psi}(x, 0\pm, s) = \tilde{\Psi}_1(x,s) + \tilde{A}_0(s) \quad (|x| < 1), \tag{29a}$$

where $$\tilde{\Psi}_1(x,s) = -\left(\frac{\partial}{\partial x} + s\right)\int_{-1}^{x} \tilde{V}(x_1,s)\,dx_1 \quad (|x| < 1), \tag{29b}$$

and $$\tilde{A}_0(s) = -s\int_{-\infty}^{-1} \tilde{v}(x, 0\pm, s)\,dx = s\int_{-\infty}^{-1} \exp\,(s(x+1))\,\tilde{\Psi}(x,0,s)\,dx. \tag{29c}$$

The last equality in (29c) is obtained readily by comparing (28b) with (28a) at $x = -1$, $y = 0$. Thus $\tilde{\Psi}$ is known except for an additive constant term $\tilde{A}_0(s)$. Furthermore, from (20) and (25) it follows that

$$\tilde{\Psi}(x, 0\pm, s) = \mathrm{Re}\,\tilde{F}(x \pm i0, s) = 0 \quad (|x| > 1). \tag{30}$$

This Riemann–Hilbert problem, specified by (29), (30), and conditions (21), (22), can be readily solved (for the general method, see, e.g., Muskhelishvili 1953, pp. 235–8), giving

$$\tilde{F}(z,s) = \frac{1}{\pi i}\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}} \int_{-1}^{1} \left(\frac{1+\xi}{1-\xi}\right)^{\frac{1}{2}} \frac{\tilde{\Psi}(\xi,0,s)}{\xi - z}\, d\xi,$$

in which the function $(z-1)^{\frac{1}{2}}(z+1)^{-\frac{1}{2}}$ is defined with a branch cut from $z = -1$ to $z = 1$, so that this function tends to 1 as $|z| \to \infty$. The leading-edge singularity can be separated out in the above solution by suitable integrations while using (29a), giving

$$\tilde{F}(z,s) = i\tilde{A}_0(s) - \frac{i}{2}\tilde{a}_0(s)\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}} + \frac{1}{\pi i}\int_{-1}^{1}\left(\frac{z^2-1}{1-\xi^2}\right)^{\frac{1}{2}} \frac{\tilde{\Psi}_1(\xi,s)}{\xi - z}\, d\xi, \tag{31a}$$

where

$$\tfrac{1}{2}\tilde{a}_0(s) = \tilde{A}_0(s) + \frac{1}{\pi}\int_{-1}^{1} \tilde{\Psi}_1(\xi,s)(1-\xi^2)^{-\frac{1}{2}}\, d\xi. \tag{31b}$$

Now, substituting the value of $\tilde{\Psi}(x,0,s)$ for $x < -1$, which can be readily deduced from (31a), into the second integral representation of (29c), then, after some appropriate integrations by parts, using (29b) and the identity

$$(x^2-1)^{-\frac{1}{2}}\frac{\partial}{\partial \xi}\frac{(1-\xi^2)^{\frac{1}{2}}}{\xi - x} = (1-\xi^2)^{-\frac{1}{2}}\frac{\partial}{\partial x}\frac{(x^2-1)^{\frac{1}{2}}}{\xi - x},$$

we determine the coefficient $\tilde{a}_0(s)$, and hence also $\tilde{A}_0(s)$ from (31b), as

$$\tilde{a}_0(s) = \frac{2}{\pi}\int_{-1}^{1} [\xi - \tilde{H}(s)(1+\xi)]\, \tilde{V}(\xi,s)(1-\xi^2)^{-\frac{1}{2}}\, d\xi, \tag{32a}$$

where

$$\tilde{H}(s) = \frac{K_1(s)}{K_0(s) + K_1(s)}, \tag{32b}$$

K_0, K_1 being the modified Bessel functions of the second kind.

After the inverse transform of (31), the solution of f becomes

$$f(z,t) = iU(t)\left[A_0(\tau) - \tfrac{1}{2}a_0(\tau)\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}}\right] + \frac{1}{\pi i}\int_{-1}^{1}\left(\frac{z^2-1}{1-\xi^2}\right)^{\frac{1}{2}}\frac{\psi_1(\xi,t)}{\xi - z}\, d\xi, \tag{33}$$

$$a_0(\tau) = -\int_0^{\tau} [b_0(\tau') + b_1(\tau')]\, H(\tau - \tau')\, d\tau' + b_1(\tau), \tag{34}$$

$$b_n(\tau) = \frac{2}{\pi}\int_0^{\pi} V(x, t(\tau)) \cos n\theta\, d\theta \quad (x = \cos\theta;\quad n = 0, 1, 2, \ldots), \tag{35}$$

$$H(\tau) = \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} \exp(s\tau)\, \tilde{H}(s)\, ds \quad (\epsilon > 0), \tag{36}$$

$$\psi_1(x,t) = -\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\int_{-1}^{x} V(\xi,t)\, d\xi, \tag{37}$$

and $A_0(\tau)$ is given by the inverse transform of (31b). In particular, the value of ϕ on the body surface is $\phi^+(x,t) = \phi(x,0+,t) = -\phi(x,0-,t) = -\phi^-(x,t)$,

$$\phi^+(x,t) = \tfrac{1}{2}U(t)\, a_0(\tau)\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}} + \frac{1}{\pi}\oint_{-1}^{1}\left(\frac{1-x^2}{1-\xi^2}\right)^{\frac{1}{2}}\frac{\psi_1(\xi,t)}{\xi - x}\, d\xi \quad (|x| < 1), \tag{38}$$

in which C over the integral sign denotes its Cauchy principal value. The first term in (38) gives the leading-edge singularity, whereas the integral term is regular wherever ψ_1 is continuous. Furthermore, $a_0(\tau)$ is the only quantity in the solution that is influenced by the history (see (34) and (37)) and requires expression in terms of τ. The pressure difference across the plate is, by (16),

$$\Delta p \equiv p^-(x,t)-p^+(x,t) = 2\rho\dot{\phi}^+(x,t) \quad (|x|<1). \tag{39}$$

The lift L acting on the plate and the moment of force, M, about the mid-cord (positive in the nose-up sense) are readily obtained by straightforward integrations as

$$L = \int_{-1}^{1} \Delta p\, dx = \pi\rho\left\{U(t)\left[a_0(\tau)-b_1(t)\right]-\frac{1}{2}\frac{d}{dt}\left[b_0(t)-b_2(t)\right]\right\}, \tag{40}$$

$$M = -\int_{-1}^{1} (\Delta p)\, x\, dx = \frac{\pi}{2}\rho\left\{U(t)\left[a_0(\tau)+b_2(t)\right]+\frac{1}{4}\frac{d}{dt}\left[b_1(t)-b_3(t)\right]\right\}. \tag{41}$$

In calculating the thrust T and energy loss E we note that the complex velocity, which can be deduced from (24) by integration to give

$$w(z,\tau) = F(z,\tau)-\int_{-\infty}^{z} F_\tau(z_1,\tau-z+z_1)\, dz_1, \tag{42}$$

has the same singularity strength at the leading edge as that of F, namely

$$w(z,\tau) \sim a_0(\tau)\left[2(z+1)\right]^{-\frac{1}{2}}+O(1) \quad \text{as} \quad |z+1|\to 0.$$

This singularity of w is known in the aerodynamic theory to give rise to a leading-edge suction (directed upstream),

$$F_s(t) = \tfrac{1}{8}\pi\rho\left[a_0(\tau)+a_0^*(\tau)\right]^2, \tag{43}$$

in which a_0^* stands for the complex conjugate of a_0.

Finally, the thrust T, power P and energy loss E can be determined in terms of $h(x,t)$ by substituting (38), (39) and (43) in (3)–(5), with ϕ, h_t, and h_x in (3)–(5) all assuming their real values with respect to t. The final result, after some manipulation, is

$$\begin{aligned} T = {} & \tfrac{1}{2}\pi\rho\{[\mathrm{Re}\,(a_0+b_0-\dot{\beta}_0)][\mathrm{Re}\,(a_0-b_1+\dot{\beta}_1)]+[\mathrm{Re}\,\dot{\beta}_0][\mathrm{Re}\,\dot{\beta}_1]\} \\ & -\frac{2}{\pi}\rho\frac{d}{dt}\int_{-1}^{1}\overset{C}{\int_{-1}^{1}}\left(\frac{1-x^2}{1-\xi^2}\right)^{\frac{1}{2}}\frac{[\mathrm{Re}\,V(x,t)]}{\xi-x}[\mathrm{Re}\,h(\xi,t)]\,d\xi\,dx, \end{aligned} \tag{44}$$

$$\begin{aligned} E = {} & \tfrac{1}{2}\pi\rho U(t)[\mathrm{Re}\,(a_0+b_0)][\mathrm{Re}\,(b_1-a_0)] \\ & +\frac{1}{\pi}\rho\frac{d}{dt}\int_{-1}^{1}\overset{C}{\int_{-1}^{1}}\left(\frac{1-x^2}{1-\xi^2}\right)^{\frac{1}{2}}\frac{[\mathrm{Re}\,V(x,t)]}{\xi-x}\left[\mathrm{Re}\int_{-1}^{\xi}V(\eta,t)\,d\eta\right]d\xi\,dx, \end{aligned} \tag{45}$$

where $\dot{\beta}_n \equiv d\beta_n(t)/dt$, and

$$\beta_n(t) = \frac{2}{\pi}\int_0^{\pi} h(x,t)\cos n\theta\, d\theta \quad (x=\cos\theta;\quad n=0,1,2,\ldots). \tag{46}$$

The power P follows simply from (6), $P = TU + E$. The manipulation involved in arriving at the above result can be considerably facilitated by making use of the following relationship.

THEOREM. *If two arbitrary functions $f(x)$, $g(x)$ and their derivatives $f'(x)$, $g'(x)$ are continuous in $-1 \leqslant x \leqslant 1$, then*

$$\int_{-1}^{1} f'(x)\,dx \oint_{-1}^{1} \left(\frac{1-x^2}{1-\xi^2}\right)^{\frac{1}{2}} \frac{g(\xi)\,d\xi}{\xi - x} = \int_{-1}^{1} g'(x)\,dx \oint_{-1}^{1} \left(\frac{1-x^2}{1-\xi^2}\right)^{\frac{1}{2}} \frac{f(\xi)\,d\xi}{\xi - x}. \tag{47}$$

This theorem can be readily proved by successive integrations by parts and by observing the identity

$$(1-\xi^2)^{-\frac{1}{2}} \frac{\partial}{\partial x}\left(\frac{(1-x^2)^{\frac{1}{2}}}{\xi - x}\right) = -(1-x^2)^{-\frac{1}{2}} \frac{\partial}{\partial \xi}\left(\frac{(1-\xi^2)^{\frac{1}{2}}}{\xi - x}\right); \tag{48}$$

the contributions from the Cauchy principal limits $\xi = x - \epsilon$ and $\xi = x + \epsilon$ cancel out as $\epsilon \to 0$.

The integrals in (44) and (45) can be converted into a series representation upon substituting for V and h their Fourier series, with their respective Fourier coefficients given by (35) and (46), into the integrand and carrying out the double integration, yielding

$$\frac{2T}{\pi\rho} = (a_0 + b_0 - \dot{\beta}_0)(a_0 - b_1 + \dot{\beta}_1) + \dot{\beta}_0 \dot{\beta}_1 - \frac{d}{dt} \sum_{n=1}^{\infty} \beta_n (b_{n-1} - b_{n+1}), \tag{44$'$}$$

$$\frac{2E}{\pi\rho} = U(t)(a_0 + b_0)(b_1 - a_0) + \frac{d}{dt} \sum_{n=1}^{\infty} \frac{1}{4n} (b_{n-1} - b_{n+1})^2, \tag{45$'$}$$

in which the real value of each of a_0, b_n, β_n is implied.

Other flow quantities of related interest are the vorticity distribution along the trailing vortex sheet and the circulation around the plate. The strength of the vorticity in dx of the vortex sheet at a point $(x, 0)$ of the wake $(x > 1)$ is $\gamma(x, t)\,dx$, positive in the counterclockwise sense, where

$$\gamma(x, t) = -2u^+(x, t), \tag{49}$$

which, by virtue of (17) and (20), satisfies the equation

$$\gamma_t + U(t)\,\gamma_x = 0, \quad \text{or} \quad \gamma_\tau + \gamma_x = 0.$$

It therefore follows that

$$\gamma(x, \tau) = \gamma(1, \tau - x + 1) = -2u^+(1, \tau - x + 1) \quad (x > 1). \tag{50}$$

By Kelvin's circulation theorem, the circulation (positive in the clockwise sense) around the plate, $\Gamma(t)$, varies at the rate

$$d\Gamma/dt = U\gamma(1, t), \tag{51}$$

which gives, upon integration,

$$\Gamma(\tau) = \int_0^\tau \gamma(1, \tau)\,d\tau = -2\int_0^t U(t)\,u^+(1, t)\,dt. \tag{52}$$

Thus, both $\gamma(x,\tau)$ and $\Gamma(\tau)$ are determined once $u^+(1,\tau)$ is found. This can be best done by evaluating first its Laplace transform $\tilde{u}^+(1,s)$. From the real part of the integral of (27) and condition (20) it follows that

$$\tilde{u}^+(1,s) = -s\,e^{-s}\int_{-1}^{1} \exp\,(sx)\,\tilde{\Phi}^+(x,s)\,dx,$$

where $\tilde{\Phi}^+(x,s)$ is the real part of $\tilde{F}(z,s)$ given by (31a) evaluated at $y = 0+$, $|x| < 1$. Substituting this expression for $\tilde{\Phi}^+(x,s)$ in the above integral, using again (47), we obtain

$$\tilde{u}^+(1,s) = \tfrac{1}{2}\pi s\,e^{-s}\,[(\tilde{a}_0+\tilde{b}_0)\,I_1(s) - (\tilde{a}_0-\tilde{b}_1)\,I_0(s)],$$

where I_n are the modified Bessel functions of the first kind. By using (32a, b) and the Wronskian $I_0(s)K_1(s)+I_1(s)K_0(s) = 1/s$, the above expression reduces to

$$\tilde{u}^+(1,s) = \tfrac{1}{2}\pi\,e^{-s}\,[\tilde{b}_0(s)+\tilde{b}_1(s)]/[K_0(s)+K_1(s)]. \tag{53}$$

$u^+(1,\tau)$ is then given by the inverse transform, which can be written as a convolution integral.

A particularly significant feature of the general solution is noteworthy at this point. If for all t

$$b_0(t)+b_1(t) = 0, \tag{54a}$$

or equivalently, if $V(x,t)$ assumes the following Fourier expansion (see (35)),

$$V(x,t) = b_0(t)\,(\tfrac{1}{2}-\cos\theta) + \sum_{n=2}^{\infty} b_n(t)\cos n\theta \quad (x = \cos\theta), \tag{54b}$$

then, according to (53), $u^+(1,t) = 0$, and hence the circulation Γ remains constant (see (51)), and the plate sheds no vorticity into the wake regardless of what values $b_0(t), b_2(t), \ldots$ may take. Thus, there are an infinite number of such modes of unsteady motion that will leave no trailing vortex sheet. Furthermore, by (34), condition (54a) also implies that $a_0(t) = b_1(t) = -b_0(t)$. It therefore follows from (44), (45) that the first terms in the expression for T and E vanish, thus leaving T, E and P to vary as the total time derivative of certain functions of t. Since no vortex sheet is shed under condition (54), those values of T, E and P can arise only from the effect of the virtual masses of the fluid. In particular, when the motion is periodic in t, the average values of T, E and P must all vanish under condition (54), implying no net transfer of momentum, no net energy loss, nor any net power required over each cycle. This last property was observed earlier by Wu (1961), and will be seen later in part 2 of this study to play a particularly significant role in the problem of the optimum shape of $h(x,t)$ for the maximum swimming efficiency.

4. Balance of recoil of a self-propelling body

When an aquatic animal propels itself along a rectilinear path, the total force and the moment of force must balance the time rate of change of their corresponding momentum. Considering the typical case of a three-dimensional planar (or slender) fish, and leaving the secondary details such as the movement of pedal fins out of the picture, it is reasonable to assume that the motive power will come only from the pure moment of internal forces that can be produced by alternating muscular contractions and relaxations. This moment is analogous to the applied bending moment in the theory of elastic beams. Whether it is possible for an aquatic animal in reality to be represented by a linear elastic body is of course quite an open question, since the elasticity of the part that is of living soft tissues has been found by Fung (1967) to be strongly non-linear. How much this will be affected by the vertebral column is still not known. We shall, however, assume that the reactions of the flexible body to the applied bending moment and hydrodynamic forces satisfy the linear elastic relationships. We shall further adopt the elementary beam theory, which is considered to be adequate here.

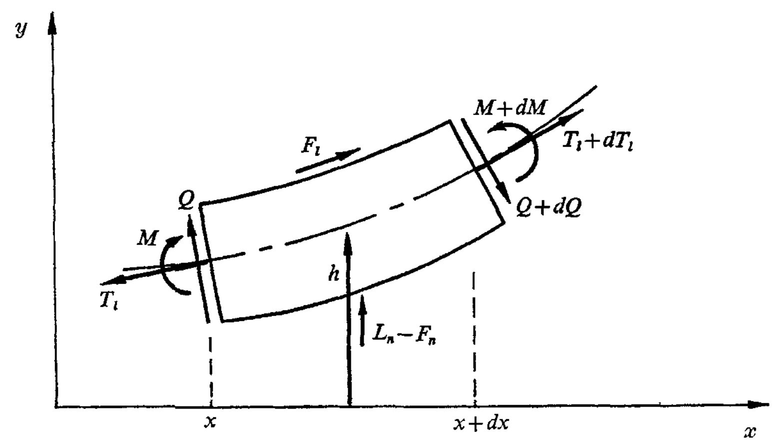


FIGURE 2. Hydrodynamic and elastic forces and moments acting on a longitudinal element of a flexible body in transverse movements. The bending moment M includes the active moment M_a due to asymmetrical muscular contractions and relaxations.

Taking the free-body diagram of a longitudinal section, of length dx, of the body (see figure 2), we obtain the equations governing the motion of a flexible body:

$$\partial T_l/\partial x + F_l = 0, \tag{55a}$$

$$L_n - F_n = m(x)\frac{\partial^2 h}{\partial t^2} + \frac{\partial Q}{\partial x} - \frac{\partial}{\partial x}\left(T_l \frac{\partial h}{\partial x}\right), \tag{55b}$$

$$Q = \frac{\partial M_a}{\partial x} + \frac{\partial}{\partial x}\left(E_y I \frac{\partial^2 h}{\partial x^2}\right), \tag{55c}$$

where $T_l(x,t)$ is the longitudinal tension induced by F_l, which represents the longitudinal component of hydrodynamic shear and pressure forces per unit length, L_n is the lift per unit length arising from the pressure, F_n denotes the transverse hydrodynamic viscous drag per unit length, m is the mass of the body per unit length, Q is the elastic shear force in the cross-sectional plane, M_a represents the applied bending moment due to muscular contractions. The quantity $(E_y I)$ stands for the effective bending rigidity, E_y being the effective Young's modulus and I the moment of inertia about the bending axis. From (55*a–c*) one can evaluate the applied moment M_a that must be required for generating the prescribed body motion h, and vice versa, with suitable end conditions (e.g. $Q = M_a = 0$ and the ends).

Qualitatively speaking, if the thrust and viscous drag are about uniformly distributed along a self-propelling body, F_l and T_l should be everywhere small. Furthermore, F_n is generally small compared with L_n at large Reynolds numbers if the cross-flow does not separate. Under these presumptions, we integrate (55*a–c*) along a slender or planar body, giving for the lift L and moment M

$$L(t) = \int_{-1}^{1} m(x)\, h_{tt}(x,t)\, dx, \tag{56a}$$

$$M(t) = -\int_{-1}^{1} xm(x)\, h_{tt}(x,t)\, dx. \tag{56b}$$

It may be remarked here that, after integration, the right-hand side of (56*b*) contains a term $E_y I h_{xx}$ evaluated at the two limits of integration, these two terms being assumed to vanish with the bending moments of inertia I at the two ends of the body. Although the integral conditions (56*a*, *b*) were given by Lighthill (1960), the set of differential equations (55*a–c*) are still thought useful for biological studies of the activating couple M_a. We further note that, although (56*a*, *b*) must be observed for all t, they are always satisfied in the mean by time harmonic motions. However, these recoil conditions, if not satisfied by a specific h of (12), may exclude the possibility of this motion being realized without extra 'rigid-body' motions of sideslip and yaw being superimposed.

It can further be remarked that, when the present two-dimensional theory is applied to evaluate the propulsion of the lunate tail of large aspect ratio, or the wing of some birds, (56*a*, *b*) need not be considered as the primary conditions for recoil balance, since this question must be settled together with the motion of the entire body.

5. Harmonic time motion with U = const.

We consider next the special case of simple harmonic motion in t, with U = const., of a two-dimensional flexible plate which starts impulsively from $h = 0$ at $t = 0$. The steady-state solution has been evaluated by Wu (1961) using the Fourier series method (compared with which the present function-theory approach seems to be simpler). We shall supplement the previous work by providing the asymptotic solutions for both large and small t.

The motion is prescribed by

$$h(x,t) = h_1(x) \exp(j\omega t) \quad (-1 \leqslant x \leqslant 1, \quad t > 0), \tag{57}$$

and $h = 0$ for $t < 0$. U = const., hence $\tau = Ut$. Then

$$V(x,t) = V_1(x) \exp(j\omega t) \quad (-1 \leqslant x \leqslant 1, \quad t > 0), \tag{58a}$$

$$V_1(x) = U\left(\frac{d}{dx} + j\sigma\right) h_1(x) \quad (\sigma = \omega/U), \tag{58b}$$

and $V = 0$ for $t < 0$. The asymptotic behaviour of the solution involves primarily the values of $a_0(t)$ for large and small t, since $a_0(t)$ is the only history-dependent term. Substituting the Laplace transform of $V(x,t)$,

$$\tilde{V}(x,s) = V_1(x)/(s-j\sigma),$$

in (32), and applying the inversion theorem to $\tilde{a}_0(s)$, we obtain

$$a_0(t) = b_1(t) - [b_0(t)+b_1(t)]\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \exp[\tau(s-j\sigma)]\,\tilde{H}(s)\frac{ds}{s-j\sigma} \quad (\epsilon > 0), \quad (59)$$

where b_0, b_1 are given by (35) (the time factor $\exp(j\omega t)$ of b_0, b_1 being recovered here) and $\tilde{H}(s)$ is given by (32b). The imaginary unit j in the above integrand can clearly be taken to be the same as $i = \sqrt{-1}$. The integrand has a simple pole at $s = i\sigma$ and a logarithmic branch point at $s = 0$, and is regular elsewhere in the finite s plane with a branch cut introduced along the negative real s-axis.

For large τ (actually large Ut/l, l being the half-chord which is being taken to be unity here), the above path of integration can be deformed into a small circle (counter-clockwise) about $s = i\sigma$ and a contour Γ circumventing counter-clockwise the entire negative real s-axis and the origin. The contour integral around $s = i\sigma$ is given immediately by the residue theorem, whereas the Γ-contour integral can be evaluated for large τ, according to Watson's lemma, by expanding the resulting integrand for small $|s|$. The final result is

$$a_0(t) = b_1 - (b_0+b_1)\,\Theta(\sigma) + \frac{(b_0+b_1)}{j\sigma\tau^2}\exp(-j\omega t)\left\{1 + O\left(\frac{1}{\tau}\log\tau\right)\right\} \quad (\tau = Ut/l \gg 1), \quad (60)$$

$$\Theta(\sigma) \equiv \frac{K_1(j\sigma)}{K_0(j\sigma)+K_1(j\sigma)} = \mathscr{F}(\sigma) + j\mathscr{G}(\sigma) \quad (\sigma = \omega l/U). \quad (61)$$

$\Theta(\sigma)$ is the Theodorsen function, $\mathscr{F}$ and $\mathscr{G}$ being its real and imaginary part respectively, and σ is the reduced frequency based on half-chord l. The last term in (60) diminishes monotonically like t^{-2} (noting that the harmonic time factors cancel out) as $t \to \infty$, yielding the steady-state solution of a_0,

$$a_0(t) = b_1 - (b_0+b_1)\,\Theta(\sigma). \quad (62)$$

For small τ, the asymptotic value of the integral in (59) can be obtained directly by expanding the integrand for large $|s|$, giving

$$a_0(t) = b_1 - \tfrac{1}{2}(b_0+b_1)\{1 + \tfrac{1}{4}\tau - \tfrac{5}{128}\tau^2 + O(\tau^3)\} \quad (\tau = Ut/l \ll 1). \quad (63)$$

This result shows that, immediately after the motion starts, the coefficient of the term (b_0+b_1) is $\frac{1}{2}$, which changes over to $\Theta(\sigma)$ as $t \to \infty$. This feature is quite similar to the Wagner function for the sharp-edged gust effect. The above asymptotic expressions for a_0, (60), (63), can be directly used to determine T, E and P at large or small values of t. It is obvious that, for t large, T, E and P differ from their respective steady-state value by a term of $O(\tau^{-2})$, which becomes negligible for $\tau > 10$, or after the body travelled over five chord lengths, provided σ is not too small.

After the transient motion falls off, the time averages of T, E and P follow immediately from (44), (45) by applying the averaging formula (11). Thus

$$\begin{aligned}\bar{T} &= \tfrac{1}{4}\pi\rho \operatorname{Re}[(a_0+b_0-\dot{\beta}_0)(a_0^*-b_1^*+\dot{\beta}_1^*)+\dot{\beta}_0\beta_1^*] \\ &= \tfrac{1}{4}\pi\rho\{|b_0+b_1|^2(\mathscr{F}^2+\mathscr{G}^2-\mathscr{F})-\operatorname{Re}(jU\sigma(b_0+b_1)[(\beta_0^*-\beta_1^*)\Theta(\sigma)+\beta_1^*])\},\end{aligned} \quad (64)$$

$$\bar{E} = \tfrac{1}{4}\pi\rho U \operatorname{Re}[(a_0+b_0)(b_1^*-a_0^*)] = \tfrac{1}{4}\pi\rho U|b_0+b_1|^2[\mathscr{F}-(\mathscr{F}^2+\mathscr{G}^2)], \quad (65)$$

$$\bar{P} = U\bar{T}+\bar{E} = \tfrac{1}{4}\pi\rho U^2 \operatorname{Re}\{-j\sigma(b_0+b_1)[(\beta_0^*-\beta_1^*)\Theta(\sigma)+\beta_1^*]\}, \quad (66)$$

the final expressions being obtained upon using (61), (62). This solution agrees with the previous result of Wu (1961, in which b_n were written as $-\lambda_n$). To this end we observe that the above $\bar{E}$ has the property $\bar{E} \not< 0$. In fact, Theodorsen's function $\Theta(\sigma) = \mathscr{F}+i\mathscr{G}$ possesses the property $\mathscr{F} \geqslant (\mathscr{F}^2+\mathscr{G}^2)$ for $\sigma \geqslant 0$, the equality holding only when $\sigma = 0$. Therefore $\bar{E} \geqslant 0$ in general, and $\bar{E} = 0$ holds either when $\sigma = 0$, a trivial case of steady motion, or when $b_0+b_1 = 0$, a special case already discussed in the sequel to (54).

6. Swimming with constant acceleration; small time optimum shape

As a typical example of variable $U(t)$ we consider the case in which a flexible plate moves forward with a constant acceleration from at rest,

$$U(t) = \alpha t \quad (\alpha > 0, \quad t > 0), \quad (67)$$

and its lateral motion is represented by a cubic form in x,

$$h(x,t) = \tfrac{1}{2}\beta_0(t)+\sum_{n=1}^{3}\beta_n(t)\cos n\theta \quad (x = \cos\theta, \quad 0 < \theta < \pi). \quad (68)$$

This profile provides enough degrees of freedom for the lateral force and angular recoil to be minimized and the optimum efficiency examined. We further assume that in a certain time interval the coefficients β_n can be expanded for small t as

$$\beta_n(t) = \sum_{m=2}^{\infty}\beta_{nm}t^m \quad (n = 0, 1, 2, 3). \quad (69)$$

This expansion starts from t^2 so that the initial plate velocity $V(x,0) = 0$. We shall, however, deal with only the first two leading terms in t, from which sufficient information can be obtained about the small time behaviour of the solution.

The corresponding Fourier series of $V(x,t)$ also has four terms,

$$V(x,t) = h_t+\alpha t h_x = \tfrac{1}{2}b_0(t)+\sum_{n=1}^{3}b_n(t)\cos n\theta. \quad (70)$$

Since h_t is of $O(t)$ whereas $\alpha t h_x$ is of $O(t^3)$, it follows that

$$b_n = \dot{\beta}_n+O(t^3) \quad (n = 0, 1, 2, 3). \quad (71)$$

To obtain the expansion of $a_0(\tau)$ for t small, we first note

$$\tau = \int_0^t U(t)\,dt = \tfrac{1}{2}\alpha t^2. \quad (72)$$

Consequently, the Laplace transform (with respect to τ) of t^m is

$$\mathscr{L}[t^m] = \int_0^\infty \exp(-s\tau)\left(\frac{2\tau}{\alpha}\right)^{\frac{1}{2}m} d\tau = \Gamma(\tfrac{1}{2}m+1)\left(\frac{2}{\alpha}\right)^{\frac{1}{2}m} s^{-(\frac{1}{2}m+1)}. \tag{73}$$

By (32),
$$\tilde{a}_0(s) = \tilde{b}_1 - (\tilde{b}_0+\tilde{b}_1)\,H(s).$$

Now, by using the asymptotic expansion of $K_n(s)$ for large $|s|$ in (32b),

$$\tilde{H}(s) = \frac{1}{2}\left[1+\frac{1}{4s}-\frac{1}{8s^2}+O(|s|^{-3})\right].$$

Hence
$$\tilde{a}_0(s) = \tfrac{1}{2}(\tilde{b}_1-\tilde{b}_0) - (1/8s)(\tilde{b}_0+\tilde{b}_1)[1+O(|s|^{-1})].$$

The Laplace inversion of the last term above is two orders smaller in t than (b_0-b_1), according to (73). Consequently,

$$a_0(t) = \tfrac{1}{2}(b_1-b_0)+O(t^3). \tag{74}$$

Next, we shall assume that the inertial mass of the thin plate is negligible and further require the resulting lift L and moment M to be small so as to eliminate the recoil. Order estimate of L and M (see (40), (41)) shows clearly

$$L = \tfrac{1}{2}\pi\rho(\dot{b}_2-\dot{b}_0)+O(t^2), \quad M = \tfrac{1}{8}\pi\rho(\dot{b}_1-\dot{b}_3)+O(t^2).$$

Thus, L and M behave like a step function across $t=0$ unless, up to $O(t^2)$,

$$b_2 = b_0, \quad b_3 = b_1; \quad \text{or} \quad \beta_2 = \beta_0, \quad \beta_3 = \beta_1. \tag{75}$$

Only under this condition will L and M vanish up to the order t^2. Under this assumption, we deduce from (44)′ and (45)′ the corresponding thrust and power required, also using (71), (74):

$$\frac{2T}{\pi\rho} = a_0^2 - \beta_1\dot{b}_0 + O(t^4), \quad \frac{2P}{\pi\rho} = \frac{2E}{\pi\rho} = \tfrac{1}{6}\,b_0\dot{b}_0 + \tfrac{1}{8}b_1\dot{b}_1 + O(t^3). \tag{76}$$

The expressions for T, P in terms of the coefficients β_{nm} are

$$2T/\pi\rho = (\beta_{02}^2+\beta_{12}^2-4\beta_{02}\beta_{12})\,t^2 + 3[\beta_{03}(\beta_{02}-3\beta_{12})+\beta_{13}(\beta_{12}-\tfrac{5}{3}\beta_{02})]\,t^3+O(t^4), \tag{77}$$

$$2P/\pi\rho = (\tfrac{2}{3}\beta_{02}^2+\tfrac{1}{2}\beta_{12}^2)\,t + 3[\beta_{02}\beta_{03}+\tfrac{3}{4}\beta_{12}\beta_{13}]\,t^2+O(t^3). \tag{78}$$

It is of interest to note that the thrust is produced by the time of $O(t^2)$, whilst the power is already required at the time of $O(t)$, the initial power being positive definite. Another point of interest is that the rectilinear acceleration α does not yet appear in the first two order terms.

A qualitative evaluation of the optimum profile at the initial stage can be made as follows. Up to the first-order terms, the thrust is maximum for fixed power P if

$$\zeta \equiv \beta_{12}/\beta_{02} = -(1+\sqrt{193})/12 = -1{\cdot}24, \tag{79}$$

which can readily be verified by the method of undetermined multipliers. Determination of the higher terms is not as simple. However, an indication of the trend of motion can still be given. If we require the second-order term of P to vanish, then the second-order term of T can be positive, under condition (79), if $\beta_{13}/\beta_{03} = 1{\cdot}07$ and β_{03}/β_{02} is sufficiently large and positive.

Under condition (75), the body has the S-shape form, the maximum and minimum of h are given by

$$\partial h/\partial x = 12\beta_1 x^2 + 4\beta_0 x - 2\beta_1 = 0,$$

or, up to the first-order term,

$$x_{1,2} = [-1 \pm (1+6\zeta^2)^{\frac{1}{2}}]/6\zeta = -0\cdot564,\ 0\cdot295. \tag{80}$$

When the higher terms in t are included, these points are seen to move back towards the trailing edge with increasing time.

Without giving the detail, the vortex sheet strength at the trailing edge is found to be

$$\gamma(1,t) = -\frac{1}{2}\frac{\pi}{\sqrt{\alpha}}\left[(\beta_{02}+\beta_{12})+\frac{6}{\pi}(\beta_{03}+\beta_{13})\,t+O(t^2)\right], \tag{81}$$

which shows that, immediately after the motion is started, there is an initial vortex shed from the trailing edge. The sense of this vortex, under condition (79), is such that the fluid near the trailing edge is propelled downstream.

I am very much indebted to Professor M. J. Lighthill for interesting and stimulating discussions, and particularly for his kindness in giving me the privilege of knowing his great work (1970) prior to its publication. This work was partially sponsored by the National Science Foundation, under Grant GK 10216, and by the Office of Naval Research, under Contract N00014-67-A0094-0012.

REFERENCES

FUNG, Y. C. 1967 *Am. J. Physiology*, **213**, 1532.

GRAY, J. 1948 *Nature, Lond.* **161**, 199.

GRAY, J. 1949 *Nature, Lond.* **164**, 1073.

GRAY, J. 1968 *Animal Locomotion.* London: Weidenfeld & Nicholson.

GRAY, J. & HANCOCK, G. J. 1955 *J. Exp. Biol.* **32**, 802.

HANCOCK, G. J. 1953 *Proc. Roy. Soc.* A **217**, 96.

JOHANNESSEN, C. L. & HARDER, J. A. 1960 *Science*, **132**, 1550.

KÁRMÁN, T. VON & BURGERS, J. M. 1943 General aerodynamic theory: perfect fluids. *Aerodynamic Theory* E2 (ed. W. F. Durand).

LANG, T. G. 1966 Hydrodynamic analysis of cetacean performance. *Whales, Dolphins and Porpoises* (ed. K. S. Norris). University of California.

LANG, T. G. & DAYBELL, D. A. 1963 *NAVWEPS Rep.* 8060; *NOTS Tech. Publ.* 3063.

LANG, T. G. & NORRIS, K. S. 1966 *Science*, **151**, 588.

LANG, T. G. & PRYOR, K. 1966 *Science*, **152**, 531.

LIGHTHILL, M. J. 1952 *Comm. Pure Appl. Math.* **5**, 109.

LIGHTHILL, M. J. 1960 *J. Fluid Mech.* **9**, 305.

LIGHTHILL, M. J. 1969 *Ann. Rev. Fluid Mech.* **1**, 413.

LIGHTHILL, M. J. 1970 *J. Fluid Mech.* **44**, 265.

MUSKHELISHVILI, N. I. 1953 *Singular Integral Equations.* Groningen, Holland: Noordhoff.

OSBORNE, M. F. M. 1960 *J. Exp. Biol.* **38**, 365.

REYNOLDS, A. J. 1965 *J. Fluid Mech.* **23**, 241.

SAFFMAN, P. G. 1967 *J. Fluid Mech.* **28**, 385.

SIEKMANN, J. 1963 *J. Fluid Mech.* **15**, 399.

STOKES, G. G. 1851 *Trans. Camb. Phil. Soc.* **9**, 8.

TAYLOR, G. I. 1951 *Proc. Roy. Soc.* A **209**, 447.

TAYLOR, G. I. 1952*a* *Proc. Roy. Soc.* A **211**, 225.

TAYLOR, G. I. 1952*b* *Proc. Roy. Soc.* A **214**, 158.

TUCK, E. O. 1968 *J. Fluid Mech.* **31**, 305.

WU, T. Y. 1961 *J. Fluid Mech.* **10**, 321.